Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 2, pp. 200–207, 2007

STABILITY OF THE EQUILIBRIUM STATE IN A CONVECTION MODEL WITH NONLINEAR TEMPERATURE AND PRESSURE DEPENDENCES OF DENSITY

V. B. Bekezhanova

UDC 532.51.013.4:536.25

The convection of a heat-conducting viscous liquid is considered. It is assumed that the liquid density depends quadratically on the temperature and pressure. The instability of the equilibrium state of a free-boundary horizontal layer with respect to small perturbations is studied using a linearization method. It is found that the state of mechanical equilibrium is unstable. Neutral curves are constructed and the critical Rayleigh numbers are found. The results are compared with the well-known solution of the same problem for the limiting case where the density is a quadratic function of temperature and does not depend on pressure.

Key words: *heat-conducting viscous liquid, free boundary, stability.*

Introduction. Observations performed at Lake Baikal indicate the presence of a deep mixing mechanism that transfers surface water to bottom regions [1]. One of the possible causes of this phenomenon is a thermal expansion anomaly.

We assume that the density is a function of only temperature and does not depend on pressure. Then, the equation of state for water becomes

$$\rho = \rho_0(1 - \alpha(\theta - \theta_0)^2), \tag{1}$$

where ρ_0 is the maximum density reached at a temperature θ_0, which is called the inversion temperature or the temperature of the thermal-expansion anomaly of the liquid, α is the thermal-expansion coefficient, and θ is the temperature. For water, $\rho_0 = 999.972$ kg/m^3, the inversion temperature is $\theta_0 = 277.13$ K, and $\alpha = 8.57 \cdot 10^{-6}$ K^{-2}. It should be noted that the maximum density is reached within the layer, i.e., the surface temperature is higher than inversion temperature, and temperature of the lower boundary is lower than the inversion temperature. In this case, a complex vertical stratification arises. In the upper part of the layer, the density increases in the direction of gravity and the liquid is gravitationally stable, and in the lower part of the layer, the liquid density stratification is unstable. The convective flows that arise in the unstable part of the liquid propagate to the upper stable zone. This phenomenon is called penetrative convection. If the thickness of the layer is insignificant, the density variation due to the pressure effect can be ignored. However, in studies of the processes occurring in deep-water pools (in particular, in Lake Baikal), it should be taken into account that pressure gradients can have a significant effect on the density distribution, and, hence, convective processes. Therefore, instead of (1) we shall use the equation of state for the liquid in the form

$$\rho = \rho_0(1 - \alpha(\theta - \theta_*)^2), \tag{2}$$

where $\theta_* = \theta_0(1 - \delta_0 p)$, p is the pressure, and ρ_0, θ_0, α, and δ_0 are positive constants. Equation (2) is a simplified version of the equation of state

$$\rho(\theta, p) = \rho_m(p)[1 - \varphi(p)(\theta - \theta_m(p))^2].$$

Institute of Computational Modeling, Siberian Division, Russian Academy of Sciences, Krasnoyarsk 660036; bekezhanova@mail.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 2, pp. 66–74, March–April, 2007. Original article submitted December 29, 2004; revision submitted May 24, 2006.

 0021-8944/07/4802-0200

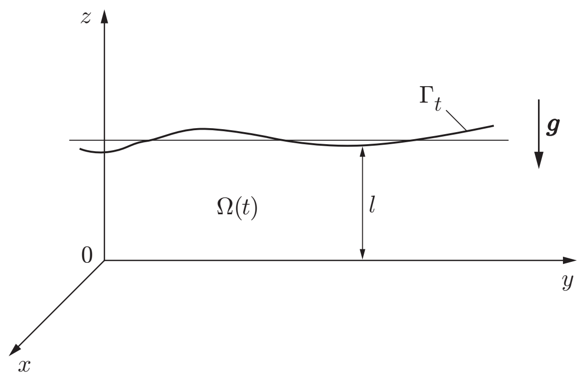

Fig. 1. Flow diagram.

The form of the functions $\rho_m(p)$, $\varphi(p)$, and $\theta_m(p)$ and the reasons for the choice of this equation of state are given in [2]. In (2), the functions $\rho_m(p)$ and $\theta_m(p)$ are replaced by the zero terms of the Taylor series of ρ_0 and θ_0, respectively. The constant δ_0 is determined from the expression for $\theta_m(p)$. For the specified values of the physical parameters (for Lake Baikal water), the error in determining the density by Eq. (2) is less than 1%.

The use of the equation of state (2) is consistent with the data of full-scale observations obtained at Lake Baikal [3, 4]. In particular, it is noted [3, 4] that the maximum density point in the lake is at a depth of 250–300 m.

In the present study, we use model equations of free convection in which thermal expansion is taken into account only in terms containing the Archimedean force (the Oberbeck–Boussinesq approximation).

1. Formulation of the Problem. Let a region $\Omega(t)$ be filled with a liquid in contact with a gas phase. The equation of state has the form (2). The x and y axes are in the plane of the lower boundary of the layer, and the z axis is directed upward. The thickness of the layer is l. The lower boundary of the layer is a solid wall, and the upper boundary is an nonderformable free boundary (Fig. 1). The surface Γ_t is defined by the equation $f(\boldsymbol{x}, t) = 0$, where $\boldsymbol{x} = (x, y, z)$. In the region, $\Omega(t)$ the Oberbeck–Boussinesq equations are valid:

$$\operatorname{div} \boldsymbol{v} = 0, \qquad \frac{\partial \theta}{\partial t} + \boldsymbol{v} \cdot \nabla \theta = \chi \Delta \theta,$$
$$\rho_0 \Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \Big) = -\nabla p + \operatorname{div}(2\mu D) + \rho \boldsymbol{g}. \tag{3}$$

Here $\boldsymbol{v} = (u, v, w)$ is the velocity, χ is the thermal diffusivity, μ is the viscosity, D is the strain rate tensor of the vector field $\boldsymbol{v}$ with the elements

$$D_{ij} = \frac{1}{2} \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \Big) \qquad (i, j = 1, 2, 3),$$

ρ is defined by formula (2), $\boldsymbol{g} = (0, 0, -g)$, and g is the acceleration of gravity.

Temperature and the attachment condition are specified at the solid wall:

$$\theta = \theta_1, \quad \boldsymbol{v} = 0 \qquad \text{at} \quad z = 0; \tag{4}$$

and kinematic, dynamic, and energetic conditions are imposed at the free surface:

$$\boldsymbol{v} \cdot \boldsymbol{n} = V_n, \qquad P \cdot \boldsymbol{n} + p_g \cdot \boldsymbol{n} = 0,$$
$$k \frac{\partial \theta}{\partial n} + b(\theta - \theta_g) = Q \qquad \text{at} \quad z = l. \tag{5}$$

Here $\boldsymbol{n}$ is the normal to the surface Γ_t, V_n is the velocity Γ_t in the direction of the normal, $P = -(p + \mu' \operatorname{div} \boldsymbol{v})I + 2\mu D(\boldsymbol{v})$ is the stress tensor in the liquid, μ' is the dilatational viscosity coefficient, I is unit tensor, k is the thermal conductivity of the liquid, b is the interfacial heat-transfer coefficient, θ_g and p_g are the gas temperature and pressure, and Q is the specified heat flux.

2. Equilibrium State. In the equilibrium state, $\theta_t = p_t = 0$ and $\boldsymbol{v}_e = 0$. The incompressibility equation is satisfied identically. From the energy equation, it follows that θ_e is a linear function z of the form

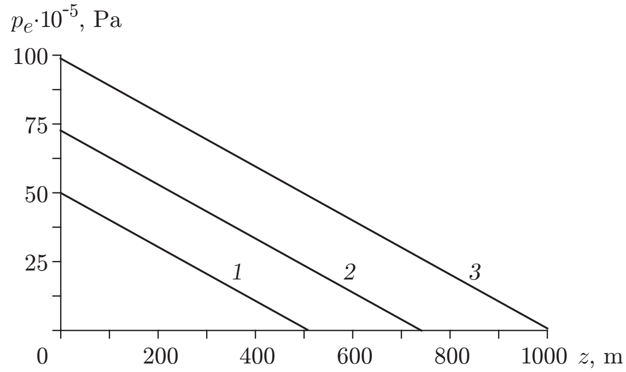

Fig. 2

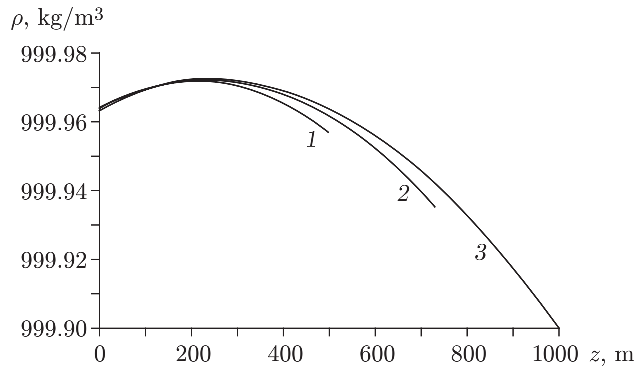

Fig. 3

Fig. 2. Curve of $p_e(z)$ for $l = 500$ (1), 730 (2), and 1000 m (3).

Fig. 3. Density distribution over the layer thickness: 1) $l = 500$ m, $l_* = 209$ m, and $\theta_0 = 3.37$°C; 2) $l = 730$ m, $l_* = 240$ m, and $\theta_0 = 2.96$°C; 3) $l = 1000$ m, $l_* = 244$ m, and $\theta_0 = 2.43$°C.

$$\theta_e(z) = Az + B, \tag{6}$$

where the constants A and B are determined from the boundary conditions at the free surface and solid wall, respectively:

$$A = \frac{Q - bB + b\theta_g}{k + bl}, \qquad B = \theta_1.$$

The momentum equation reduces to the equation

$$p_z = -\rho g. \tag{7}$$

We denote $p_e = p_1 + \rho_0 \boldsymbol{g} \cdot \boldsymbol{x} = p_1 - \rho_0 g z$. Then, Eq. (7) becomes

$$p_{1z} = C(p_1 + Dz + E),$$

where

$$C = \rho_0 g \alpha \theta_0^2 \delta_0^2 > 0, \qquad D = \frac{A - \theta_0 \delta_0 g \rho_0}{\theta_0 \delta_0} < 0, \qquad E = \frac{B - \theta_0}{\theta_0 \delta_0} < 0.$$

The constants D and E are negative for the real physical parameters of the liquid (Lake Baikal water).

Setting $Dz + E = \eta$, $p_1 + \eta = y$, and $C/D = C_1$, we obtain the equation

$$\frac{dy}{d\eta} = 1 + C_1 y^2.$$

After the change $C_1 y^2 = -x^2$ (since $C_1 < 0$) inverse substitutions, solution (7) is written as

$$p_e = \frac{1}{\sqrt{|C_1|}} \frac{C_3 H(z) - 1}{C_3 H(z) + 1} - Dz - E - \rho_0 g z,$$

where the constant C_3 is determined from the dynamic condition at the free boundary:

$$C_3 = \frac{1 + \sqrt{|C_1|}\,(p_g + Dl + E + \rho_0 g l)}{H(l)\sqrt{|C_1|}\,(p_g + Dl + E + \rho_0 g l)},$$

$H(z) = \exp(2\sqrt{|C_1|}\,(Dz + E))$.

The function p_e is downward convex and nearly linear (Fig. 2). The lower boundary corresponds to the value $z = 0$.

Substitution of p_e into (2) into yields the function $\rho(z)$. The curves of $\rho(z)$ plotted for various layer thicknesses l are shown in Fig. 3 [l_* is the layer thickness for which the water density ρ takes the maximum value (the coordinate of the inversion].

Thus, we obtained a steady-state solution $\boldsymbol{v}_e$, p_e, θ_e of the boundary-value problems (3)–(5) that corresponds to the state of mechanical equilibrium.

3. Dimensionless Parameters. We write system (3) in dimensionless variables. For this, we use the width l_* of the lower part of the layer, in which the liquid is stratified unstably, as the characteristic length, the difference $T = \theta_1 - \theta_0$ as the temperature scale, and the velocity of convective rise of a heated liquid particle $v_* = \sqrt{gl_*\alpha T^2}$ as the velocity scale. For the density and pressure, we use the sales ρ_0 and $\rho_0 v_*^2$, respectively. The temperature is reckoned from the temperature of the lower boundary θ_1, and the pressure from the hydrostatic pressure.

We introduce the dimensionless variables $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ and τ such that

$$\boldsymbol{x} = (x, y, z) = \boldsymbol{\xi} l_*, \quad t = \frac{l_*}{v_*}\,\tau, \quad l_* = \frac{l}{\lambda}, \quad \lambda = \frac{\theta_1 - \theta_b}{T},$$

$$p = \rho_0 v_*^2 p', \quad \boldsymbol{v} = v_* \boldsymbol{v}', \quad \theta = T\theta'.$$

Here λ is the inversion parameter, θ_b is the free-surface temperature calculated by formula (6), p', v', and θ' are dimensionless functions of the pressure, velocity, and temperature, respectively.

Under the adopted assumptions, the free-convection equations in the dimensionless variables are written as follows (the primes are omitted):

$$\begin{gathered}\operatorname{div}\boldsymbol{v} = 0, \qquad \frac{\partial\theta}{\partial t} + \boldsymbol{v}\cdot\nabla\theta = \delta\Delta\theta, \\ \frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \mu_1\Delta\boldsymbol{v} - \Big(\frac{1}{\beta} - (\theta + \varepsilon_T p)^2\Big)\boldsymbol{k}.\end{gathered} \tag{8}$$

Here $\delta = \chi/(l_* v_*)$ is the Fourier number, $\mu_1 = \nu/(l_* v_*)$ is the kinematic-viscosity parameter (the reciprocal of the Reynolds number), $\nu = \mu/\rho_0$ is the kinematic viscosity, $\beta = \alpha T^2$, $\varepsilon_T = \theta_0\delta\rho_0 v_*^2/T$, and $\boldsymbol{k}$ is the unit vector of the z axis.

The boundary conditions in the dimensionless variables become

$$\theta = 0, \quad \boldsymbol{v} = 0 \qquad \text{at} \quad \zeta = 0; \tag{9}$$

$$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \quad \frac{\partial\theta}{\partial\zeta} + \mathrm{Bi}\,(\theta - \theta_g) = Q, \quad p = 0 \qquad \text{at} \quad \zeta = \lambda. \tag{10}$$

Here $\mathrm{Bi} = bl_*/k$ is the Biot number and $Q = kT/l_*$ is the dimensionless heat flux.

4. Small-Perturbation Equations. Let $\boldsymbol{v}_d(\boldsymbol{\xi},\tau) = \boldsymbol{v}(\boldsymbol{\xi},\tau) + \delta\boldsymbol{V}(\boldsymbol{\xi},\tau)$, $p_d(\boldsymbol{\xi},\tau) = p(\boldsymbol{\xi},\tau) + \mu_1\delta P(\boldsymbol{\xi},\tau)$, and $\theta_d(\boldsymbol{\xi},\tau) = \theta(\boldsymbol{\xi},\tau) + \Theta(\boldsymbol{\xi},\tau)$, where $\boldsymbol{V} = (U, V, W)$, P and Θ are perturbations, and $\boldsymbol{v}$, p, θ is the basic solution. The form of the functions $\boldsymbol{v}_d$, p_d, and θ_d describing the perturbed motion is chosen to simplify the subsequent transformations. The functions $\boldsymbol{v}_d$, p_d, and θ_d are solutions of Eqs. (8) subject to boundary conditions (9) and (10).

The linearized system has the form

$$\begin{gathered}\operatorname{div}\boldsymbol{V} = 0, \qquad \frac{\partial\Theta}{\partial\tau} + \boldsymbol{v}\cdot\nabla\Theta + \delta\boldsymbol{V}\cdot\nabla\theta = \delta\Delta\Theta, \\ \delta\,\frac{\partial\boldsymbol{V}}{\partial\tau} + \delta(\boldsymbol{V}\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{V}) = -\mu_1\delta\nabla P + \mu_1\Delta\boldsymbol{v} + \mu_1\delta\Delta\boldsymbol{V} - 2(\theta + \varepsilon_T p)(\Theta + \mu_1\delta\varepsilon_T P)\boldsymbol{k}.\end{gathered} \tag{11}$$

The equations of system (11) are valid in the region Ω. At the solid wall, the following conditions are satisfied:

$$\boldsymbol{V} = 0, \qquad \Theta = 0. \tag{12}$$

The conditions at the free boundary are written as follows [5]:

$$\begin{gathered}F_{1\tau} + \boldsymbol{v}\cdot\nabla F_1 + \delta\boldsymbol{V}\cdot\nabla f_1 = 0, \\ -\mu_1\delta P + 2\mu_1\delta D(\boldsymbol{V})\boldsymbol{n}\cdot\boldsymbol{n} + 4\mu_1 D(\boldsymbol{v})\boldsymbol{n}\cdot\boldsymbol{n}_1 = \frac{\partial p}{\partial n}\,R - 2\mu_1\,\frac{\partial D(\boldsymbol{v})}{\partial n}\,\boldsymbol{n}\cdot\boldsymbol{n}R, \\ \delta D(\boldsymbol{V})\boldsymbol{n}\cdot\boldsymbol{x}_{\alpha_{1,2}} + \frac{\partial D(\boldsymbol{v})}{\partial n}\,\boldsymbol{n}\cdot\boldsymbol{x}_{\alpha_{1,2}}R + D(\boldsymbol{v})\boldsymbol{n}\cdot(R\boldsymbol{n})_{\alpha_{1,2}} + D(\boldsymbol{v})\boldsymbol{n}_1\cdot\boldsymbol{x}_{\alpha_{1,2}} = 0,\end{gathered} \tag{13}$$

$$\Big(\frac{\partial\Theta}{\partial n}+\frac{\partial^2\theta}{\partial n^2}R+\nabla\theta\cdot\boldsymbol{n}_1\Big)+\mathrm{Bi}\Big(\Theta+\frac{\partial\theta}{\partial n}R\Big)=0.$$

Here $F_1=F_1(\boldsymbol{\xi},\tau)$ is the perturbation f_1; $f_1=f_1(\boldsymbol{\xi},\tau)=\zeta-f(\xi,\eta,\tau)=0$ is the equation of the unperturbed boundary, R is the local departure of the free boundary along the normal from the unperturbed state; and $\boldsymbol{n}_1$ is the perturbation of the normal $\boldsymbol{n}$:

$$\boldsymbol{n}_1=\frac{1}{EG-F^2}\,[(FR_{\alpha_2}-GR_{\alpha_1})\boldsymbol{x}_{\alpha_1}+(FR_{\alpha_1}-ER_{\alpha_2})\boldsymbol{x}_{\alpha_2}],$$

$E=|\boldsymbol{x}_{\alpha_1}|^2$, $G=|\boldsymbol{x}_{\alpha_2}|^2$ and $F=\boldsymbol{x}_{\alpha_1}\cdot\boldsymbol{x}_{\alpha_2}$ are the coefficients of the first quadratic formula, and $\boldsymbol{x}(\alpha_1,\alpha_2,\tau)$ is the free surface Γ_t specified in parametric form. In the case considered, $\boldsymbol{x}_{\alpha_1}=(1,0,0)$, $\boldsymbol{x}_{\alpha_2}=(0,1,0)$, $\boldsymbol{n}=(0,0,1)$.

System (11) is supplemented by the initial conditions

$$\boldsymbol{V}=\boldsymbol{V}_0(\boldsymbol{\xi}),\qquad \operatorname{div}\boldsymbol{V}_0(\boldsymbol{\xi})=0,\qquad \Theta=\Theta(\zeta).$$

5. Problem of Small Perturbations of Equilibrium. We consider the problem (11)–(13) of the equilibrium of a free-boundary layer described by the functions $\boldsymbol{v}_e$, p_e, and θ_e. The system of equations for the perturbations in dimensionless coordinates is written as

$$U_\xi+V_\eta+W_\zeta=0,\qquad \Theta_\tau+\delta h_1W=\delta\Delta\Theta,$$

$$U_\tau/\mu_1=-P_\xi+\Delta U,\qquad V_\tau/\mu_1=-P_\eta+\Delta V,\tag{14}$$

$$W_\tau/\mu_1=-P_\zeta+\Delta W+\mathrm{R}\,(\theta+\varepsilon_Tp)\Theta+2(\theta+\varepsilon_Tp)\varepsilon_TP,$$

where $h_1=Al_*/T$; $\mathrm{R}=2/(\mu_1\delta)$ is the Rayleigh number.

The boundary conditions are given by

$$U=V=W=0,\quad \Theta=0\qquad\text{at}\quad \zeta=0;\tag{15}$$

$$\begin{gathered}-R_\tau+\delta W=0,\qquad U_\zeta+W_\xi=0,\qquad V_\zeta+W_\eta=0,\\ -\mu_1\delta P+2\mu_1\delta W_\zeta=h_2R,\qquad \Theta_\zeta+\mathrm{Bi}\,(\Theta+h_1R)=0\qquad\text{at}\quad\zeta=\lambda,\end{gathered}\tag{16}$$

where $h_2=\partial p/\partial\zeta$.

The solution of the boundary-value problem (14)–(16) is sought for in the form of normal waves:

$$(\boldsymbol{V},P,\Theta,R)=(\boldsymbol{V}(\zeta),P(\zeta),\Theta(\zeta),R(\zeta))\exp\,[i(\alpha_1\xi+\alpha_2\eta-C\tau)].\tag{17}$$

Here α_1 and α_2 are the dimensionless wavenumbers along the x and y axes, respectively, and $C=C_r+C_i$ is a complex decrement that defines the perturbation propagation in time. Substitution of (17) into (14)–(16) yields a problem for which we can use the Squire transformation $Z=i\alpha_1U+i\alpha_2V$. After the transformation, the system becomes

$$Z+W'=0,\qquad -iC\Theta+\delta h_1W=\delta(\Theta''-k^2\Theta),$$

$$-iCZ/\mu_1=k^2P+Z''-k^2Z,\tag{18}$$

$$-iCW/\mu_1=-P'+W''-k^2W+\mathrm{R}\,(\theta+\varepsilon_Tp)\Theta+2(\theta+\varepsilon_Tp)\varepsilon_TP,$$

where $k^2=\alpha_1^2+\alpha_2^2$ is the modified wavenumber. The boundary conditions are written as

$$Z=0,\quad W=0,\quad \Theta=0\qquad\text{at}\quad\zeta=0;\tag{19}$$

$$Z'=0,\quad \Theta'+\mathrm{Bi}\,(\Theta+h_1\delta iW/C)=0,\quad -P+2W'=\mathrm{R}\,h_2iW/(2C)\qquad\text{at}\quad\zeta=\lambda.\tag{20}$$

The boundary-value problem (18)–(20) is an eigenvalue problem for the complex decrement C. For the equilibrium state $\boldsymbol{v}_e$, p_e, θ_e to be stable with respect to small perturbations of the form (17), it is necessary and sufficient that the imaginary part C_i of all eigenvalues C be negative.

6. Asymptotic Behavior of Long Waves. The parameters Z, W, P, Θ, and C can be written as follows (for $k \to 0$):

$$Z = Z_0 + kZ_1 + \ldots, \qquad W = W_0 + kW_1 + \ldots, \qquad P = P_0 + kP_1 + \ldots,$$

$$\Theta = \Theta_0 + k\Theta_1 + \ldots, \qquad C = C_0 + kC_1 + \ldots .$$

After substitution of the indicated expansion into system (18), we write the resulting equation in a zero approximation

$$Z_0'' = -iC_0 Z_0/\mu_1 \tag{21}$$

with the boundary conditions

$$Z_0 = 0 \quad \text{at} \quad \zeta = 0, \qquad Z_0' = 0 \quad \text{at} \quad \zeta = \lambda. \tag{22}$$

Multiplying Eq. (21) by the complex-conjugate value Z_0^* and integrating the result over the segment $[0, \lambda]$, we obtain

$$\int\limits_0^\lambda |Z_0'|^2 \, d\xi = -\frac{iC_0}{\mu_1} \int\limits_0^\lambda |Z_0|^2 \, d\xi,$$

whence $-iC/\mu_1 > 0$. Because $\mu_1 > 0$, it follows that $-iC > 0$. Therefore, $C_0 = iC_i$ is a purely imaginary number and $C_i < 0$. This implies that the long-wave perturbations damp monotonically.

Let us refine the form of C_0. We denote $iC_0/\mu_1 = \mu_2$. Then, Eq. (21) can be written as

$$Z_0'' + \mu_2 Z_0 = 0.$$

Since $\mu_2 > 0$, we have $Z_0 = C_1 \cos \sqrt{\mu_2}\, \zeta + C_2 \sin \sqrt{\mu_2}\, \zeta$. In the last expression, the constants C_1 and C_2 are determined from the boundary conditions (22). In this case, $C_1 = 0$, $\mu_2 = \pi^2 n^2$, and

$$C_0 = -i\mu_1 \pi^2 n^2. \tag{23}$$

7. Numerical Solution. The spectral problem (18)–(20) is solved using an orthogonalization method [6]. System (18) is brought to the form $y' = Ay$, where $y(x)$ is the vector of unknown quantities and $A(x)$ is the coefficient matrix; and $0 \leqslant x \leqslant 1$. After the substitution

$$x = \zeta/\lambda, \quad y_1 = Z, \quad y_2 = Z', \quad y_3 = W, \quad y_4 = P, \quad y_5 = \Theta, \quad y_6 = \Theta'$$

we obtain the problem

$$y_1' = \lambda y_2, \qquad y_2' = (-iC\lambda/\mu_1 + k^2\lambda) y_1 - k^2 \lambda y_4, \qquad y_3' = -\lambda y_1,$$

$$y_4' = -\lambda^3 y_2 + \lambda(iC/\mu_1 - k^2) y_3 + 2(\theta + \varepsilon_T p)\varepsilon_T \lambda y_4 + \lambda \,\mathrm{R}\, (\theta + \varepsilon_T p) y_5, \tag{24}$$

$$y_5' = \lambda y_6, \qquad y_6' = h_1 \lambda y_3 + \lambda(k^2 - iC/\delta) y_5$$

with the boundary conditions

$$y_1 = 0, \quad y_3 = 0, \quad y_5 = 0 \qquad \text{at} \quad x = 0,$$

$$y_2 = 0, \quad 2\lambda y_1 + \mathrm{R}\, h_2 i y_3/(2C) + y_4 = 0, \quad \mathrm{Bi}\, h_1 \delta i y_3/C + \mathrm{Bi}\, y_5 + y_6 = 0 \quad \text{at} \quad x = 1.$$

For $x = 1$, the boundary conditions can be written in the matrix form $Dy(1) = 0$, where D is a 3×6 matrix, whose nonzero elements have the following values:

$$d_{12} = d_{24} = d_{36} = 1, \quad d_{21} = 2\lambda, \quad d_{23} = \mathrm{R}\, h_2 i/(2C), \quad d_{33} = \mathrm{Bi}\, h_1 \delta i/C, \quad d_{35} = \mathrm{Bi}\,.$$

The remaining elements of the matrix D are equal to zero.

The solution of system (24) is sought in the form

$$y = \sum_{j=1}^{3} p_j y^j,$$

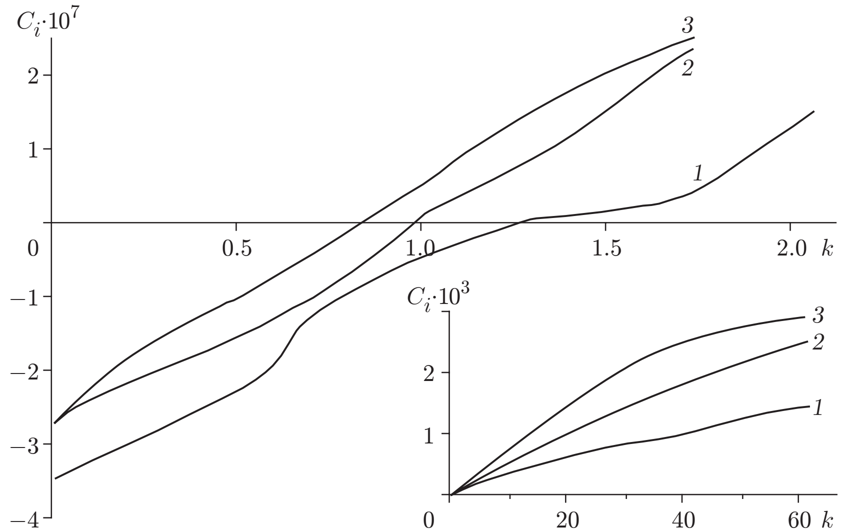

Fig. 4

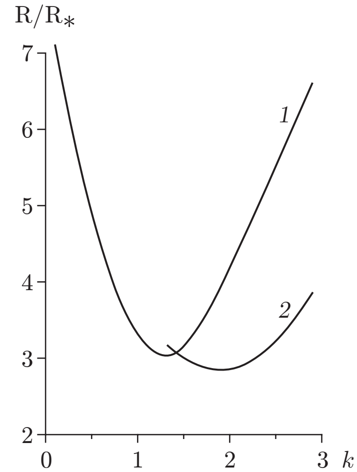

Fig. 5

Fig. 4. Complex decrements $C_i(k)$: 1) $l = 500$ m, $\mathrm{R} = 1.51 \cdot 10^{16}$, $\mathrm{Bi} = 0.093$, and $k_* = 1.31$; 2) $l = 730$ m, $\mathrm{R} = 2.72 \cdot 10^{16}$, $\mathrm{Bi} = 0.107$, and $k_* = 0.984$; 3) $l = 1000$ m, $\mathrm{R} = 2.61 \cdot 10^{16}$, $\mathrm{Bi} = 0.109$, and $k_* = 0.829$.

Fig. 5. Neutral curves $\mathrm{R}(k)$: 1) $l = 1000$ m, $\mathrm{R}_1 = 3578.3$, $k_1 = 1.42$, $\mathrm{R}_* = 1178.5$, and $k_* = 2.03$; 2) $l = 500$ m, $\mathrm{R}_1 = 3280.2$, $k_1 = 1.98$, $\mathrm{R}_* = 1178.5$, and $k_* = 2.03$.

where the coefficients p_j are found from the system $Dy(1) = 0$ and y^1, y^2, and y^3 are linearly independent vectors:

$$y^1(0) = (0, 1, 0, 0, 0, 0), \quad y^2(0) = (0, 0, 0, 1, 0, 0), \quad y^3(0) = (0, 0, 0, 0, 0, 1).$$

To find the eigenvalue C, it is necessary to know the initial approximation C_0, which is chosen from condition (23).

The stability of a free-boundary water layer was studied for the following parameter values: $\delta_0 = 5 \cdot 10^{-8}$ Pa^{-1}, $\theta_g = 291$ K, $p_g = 101{,}300$ Pa, $\nu = 1.45 \cdot 10^{-6}$ m^2/sec, and $\chi = 1.32 \cdot 10^{-7}$ m^2/sec. The calculations were performed for $l = 500$, 730, and 1000 m. The value $l = 730$ m corresponds to the average depth of Lake Baikal. For the indicated values of the physical parameters, the dependence of $C_i = \mathrm{Im}\, C$ on the wavenumber k was obtained.

Figure 4 gives curves of $C_i(k)$ for various layer thicknesses l (k_* are the critical wavenumbers for which the equilibrium state becomes unstable). The same curve for a wider range of the wavenumber k is shown in the insert.

8. Comparison of Results. The stability boundary is determined from the relation $C_i(\mathrm{R}) = 0$. Neutral perturbations correspond to the case $C_i = 0$. Setting $C = 0$ in the problem (18)–(20), we obtain the neutral curves of stability. Let us compare the results obtained with the known results of the same problem for the limiting case where the equation of state has the form of (1). This problem is considered in [7]. Figure 5 shows a curve of the Rayleigh number versus the wavenumber (neutral curves). The ordinate shows the ratio of the Rayleigh number obtained in the present study to the critical values of the Rayleigh number R_* determined in [7] (R_1 are the critical Rayleigh numbers which are the minimum values on the corresponding neutral curves; k_1 are the critical wavenumbers for which the values of R_1 are reached). In Fig. 5 it is evident that the values of R_* are much smaller than the values of R_1. In addition, it should be noted that as the Biot number decreases, the critical values of the Rayleigh number decrease and the instability region is shifted to the larger values of the wavenumber.

I thank V. K. Andreev for the useful discussions.

This work was supported by the Krasnoyarsk Regional Science Foundation (Grant No. 12F003M) and Integration project of the Siberian Division of the Russian Academy of Sciences No. 131 and the Program of Support of Leading Scientific Schools of the Russian Federation (Grant No. NSh-5873.2006.1).

REFERENCES

1. M. N. Shimaraev and N. G. Granin, “On the stratification and convection mechanism in Baikal,” *Dokl. Akad. Nauk SSSR*, **321**, No. 2, 381–385 (1991).
2. O. B. Bocharov, O. F. Vasilev, and T. É. Ovchinnikova, “Approximate equation of state for fresh water near the temperature of maximum density,” *Izv. Ross. Akad. Nauk, Fiz. Atmos. Okeana*, **35**, No. 4, 556–558 (1999).
3. T. M. Ravens, O. Kosis, A. Wuest, and N. G. Granin, “Small-scale turbulence and vertical mixing in Lake Baikal,” *Limnol. Oceanogr.*, No. 45, 159–173 (2000).
4. M. N. Shimaraev, V. I. Verbolov, N. G. Granin, and P. P. Sherstyankin, *Physical Limnology of Lake Baikal: A Review*, S. n., Irkutsk–Okayama (1994).
5. V. K. Andreev, “Small perturbations of thermocapillary liquid flow with an interface,” in: *Mathematical Modeling in Mechanics*, Proc. Workshop, Vol. 1, Institute of Computational Simulation, Sib. Div., Russian Academy of Sciences, Krasnoyarsk (1997), pp. 27–40. (Deposited in VINITI 02.12.97, No. 446-1397.)
6. S. K. Godunov, “On the numerical solution of boundary-value problems for systems of linear ordinary differential equations,” *Usp. Mat. Nauk*, **16**, No. 3, 171–174 (1961).
7. K. A. Nadolin, “Convection in a horizontal liquid layer with inversion of specific volume,” *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 1, 43–49 (1989).